Engineering Notes

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Least Acceleration Motion for Given Terminal Conditions

Vibeke Raae*
University of Trondheim, Trondheim, Norway

MOTION of a particle from a given initial position P_i to a given final position P_f with given terminal velocities v_i and v_f and given transfer time $t_f - t_i$ may be performed in infinitely many ways. How do we choose the motion requiring the least upper limit of acceleration magnitude needed?†

In Fig. 1 the initial velocity vector $\boldsymbol{v}_i$, the final velocity vector $\boldsymbol{v}_f$, and the mean velocity vector $\boldsymbol{v}_m$, all originating at P_i, define a fixed plane π_0. Another plane π (not shown) parallel to π_0 contains the terminus of the vector $(t-t_i)\boldsymbol{v}_m$. Moving with the constant mean velocity $\boldsymbol{v}_m$, the plane π defines an inertial two-dimensional coordinate system within which the optimal particle motion takes place, simply because additional motion normal to the plane π would increase the acceleration magnitude. Our problem is, hence, reduced from three- to two-dimensional. Choosing the origin at the terminus of the vector $(t-t_i)\boldsymbol{v}_m$, we have a particle motion in the plane π starting from the origin with the velocity $\boldsymbol{v}_i - \boldsymbol{v}_m$ and returning to the origin with the velocity $\boldsymbol{v}_f - \boldsymbol{v}_m$. Hence it follows that

$$\int_{t_i}^{t_f} (\boldsymbol{v} - \boldsymbol{v}_m)\,\mathrm{d}t = 0$$

i.e., the point corresponding to the "center of mass" of the hodograph must be situated at the origin, $\mathrm{d}t$ now playing the role of mass element of the hodograph curve.

The case of even mass distribution now naturally recommends itself, and leads us to consider the case when the acceleration magnitude is constant during the motion. It follows easily that a catenary shaped hogograph then becomes optimal. This is due to the fact that the center of mass of a suspended chain of a certain length is lower for the catenary equilibrium configuration than for any other configuration. Hence the "shortest" hodograph curve in question is catenary shaped, and the acceleration magnitude being equal to "curve length" divided by transfer time is minimized.

Further use of the chain analogy shows that a nonconstant magnitude of acceleration may be excluded as the optimal possibility. Considering the fact that a chain of given mass with uneven mass distribution lowers its center of mass if its thicker parts are stretched out to reduce mass distribution, this becomes more or less evident.

The above considerations for motion relative to an inertial frame are easily generalized to the case of a constant gravitational field, as, e.g., when considering the maneuver of a combat aircraft. To compensate for gravitation, we simply move the final position to a point $\frac{1}{2}g(t_f - t_i)^2$ above P_f and change the final velocity from $\boldsymbol{v}_f$ to $\boldsymbol{v}_f - \boldsymbol{g}(t_f - t_i)$.

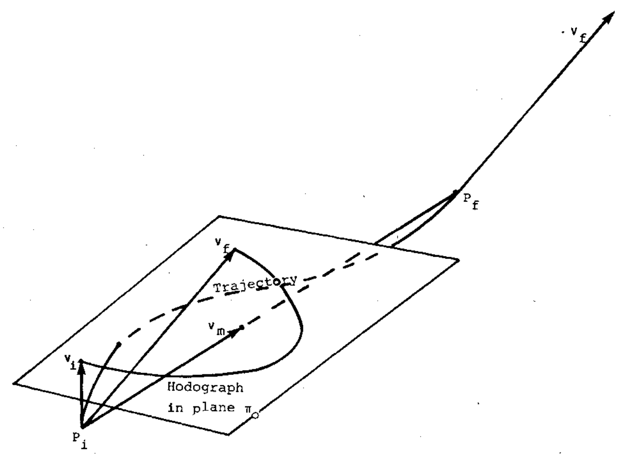

Fig. 1 Least accelerated motion with given terminals.

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*Research Student.

†This problem was raised and solved in a particular case by Harald B. Klepp of the Royal Norwegian Naval Academy. The solution in the general case was conjectured by Mari Vårhus of the University of Trondheim, Norway.

Separation of Time Scales in Aircraft Trajectory Optimization

M.D. Ardema*
NASA Ames Research Center
Moffett Field, California
and
N. Rajan†
Stanford University, Stanford, California

Introduction

SINGULAR perturbation methods have been shown by many authors[1-19] to be effective and efficient in the analysis and computation of aircraft optimal trajectories because they reduce the order of individual integrations. Singular perturbation methods depend upon identifying the characteristic "small" parameters of the dynamical systems or, equivalently, upon identifying the characteristic time scales of the state variables. Several authors[4,15] have observed that there is presently no method of casting complex (high-order, highly coupled, highly nonlinear) aircraft trajectory optimization problems in a singular perturbation form which is both rigorous and practical, and that this constitutes a gap

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*Research Scientist. Associate Fellow AIAA.

†Research Associate. Member AIAA.

in the systematic application of singular perturbation methods.

Three approaches to the problem of time-scale selection in nonlinear dynamical systems have been suggested: 1) direct identification of small parameters, usually through the use of nondimensionalization; 2) transformation of state equations; and 3) linearization of state equations. Although suitable small parameters have been identified in certain simple flight mechanics problems,[20] the unsuccessful efforts of several investigators to find such parameters for the more complex problems of primary interest have turned attention to other approaches.

Kelley[4] has considered transformations of state variables for nonlinear systems that reduce system coupling and expose time-scale characteristics. The transformations involved, however, are given by complex partial differential equations, making this approach generally impractical for complex systems.

The third possibility is to linearize the nonlinear state equations and then analyze the eigenvalues of the linearized system for time-scale separation properties. Analysis of time-scale separation in linear systems has been discussed by Chow and Kokotovic[21] and Syrcos and Sannuti.[22] This apporach, however, has drawbacks: it assumes that an optimal trajectory of the "exact" system is known (around a point of which the linearization is to be performed) and the time-scale properties will be valid only locally in the vicinity of the point about which the linearization is performed.

Because of the difficulties inherent in the above approaches, almost all singular perturbation analyses of aircraft trajectory optimization have relied on an ad hoc selection of time scales based largely on physical insight and past experience with similar problems. Clearly, what is needed is a rational method of identifying time-scale separations that requires a knowledge only of the state equations and, preferably, is globally valid. It is the purpose of this Note to suggest and demonstrate such methods.

Estimation of State Variable Speed

We consider dynamical systems of the form

$$\dot{x}=f(x,u), \qquad u\in U \tag{1}$$

subject to suitable boundary conditions where x is an n-dimensional state vector, u an r-dimensional control vector, and U the set of admissible controls. It is assumed that bounds have been established on the components of the state vector, either by physical limitations or by a desire to restrict the state to a certain region of state space,

$$x_m \leq x \leq x_M, \qquad \Delta x = x_M - x_m \tag{2}$$

Table 1 Time-scale assumptions used or proposed in past analyses of aircraft trajectory optimization by singular perturbation methods

Problem	Reference	Variable ordering: Slowest				Fastest
Three-dimensional flight	1,3,4	(x,y)		(E,χ)		(h,γ)
Three-dimensional flight	2,4	(x,y,χ)				(E)
Minimum time-to-climb in vertical plane	5-7	(E)				(h,γ)
Three-dimensional minimum time maneuvers	8	(χ)		(E)		(h,γ)
Minimum time-to-climb in vertical plane	9,10	(E)		(h)		(γ)
Three-dimensional weapon delivery	11	(x,y,E)	(χ)		(h)	(γ)
Three-dimensional interception	12	(x,y)	(E)	(χ)	(h)	(γ)
Interception in horizontal plane	13,14	(x,y)		(χ)		(V)
Three-dimensional flight	15	(x,y)	(E)		(h)	(γ,χ)
		(x,y)	(E)	(χ)	(h)	(γ)
		(x,y,E,χ)		(h)		(γ)
Pursuit/evasion in horizontal plane	17,18	(x,y)		(V)		(χ)
Pursuit/evasion in vertical plane	19	(x,y)	(E)		(h)	(γ)

Table 2 Relations for state variable speeds for aircraft trajectory optimization according to Eqs. (4) and (5)

State variable	Method 1, Eq. (4): Speed S_1	Method 1, Eq. (4): Flight condition	Method 2, Eq. (5): Speed S_2	Method 2, Eq. (5): Control variables
x,y	V_M/R	Horizontal straight flight a maximum velocity	$\bar{V}\cos\bar{\gamma}\cos\bar{\chi}/\Delta x$, $\bar{V}\cos\bar{\gamma}\sin\bar{\chi}/\Delta y$	–
h	$V_M/\Delta h$	Vertical climb at maximum velocity	$\bar{V}\sin\bar{\gamma}/\Delta h$	–
V	$\dfrac{g}{\Delta V}\left[\max\limits_{h,V}(T_M-D_0)+1\right]$	Vertical dive at maximum (thrust-drag) with maximum throttle and zero lift	$\dfrac{g}{\Delta V}(\bar{T}_M-\bar{D}_0-\sin\bar{\gamma})$	Maximum throttle and zero lift
χ	∞	Vertical flight	$\dfrac{g\bar{L}_M}{\bar{V}\cos\bar{\gamma}\Delta\chi}$	90 deg bank and maximum lift
γ	$\dfrac{gn}{\min\limits_h(V_c)\Delta\gamma}$	Vertical flight at minimum corner velocity, zero bank, and maximum load factor	$\dfrac{g}{\bar{V}\Delta\gamma}(\bar{L}_M-\cos\bar{\gamma})$	Zero bank and maximum lift
E	$\max\limits_{h,V}[V(T_M-D_0)]/\Delta E$	Flight at maximum velocity (thrust-drag) with maximum throttle and zero lift	$\bar{V}(\bar{T}_M-\bar{D}_0)/\Delta E$	Maximum throttle and zero lift

In many cases, these bounds may be necessarily arbitrary; however, since the main interest is in identifying large differences in time scales, such estimates can be relatively crude.

Most ad hoc assessments of time-scale separation are based on the concept of state variable "speed."[15,23] The speed of a state variable is defined as the inverse of the time it takes that variable to change across a specified range of values; this can be expressed by

$$S_i = \frac{\dot{x}_i}{\Delta x_i} = \frac{f_i(x,u)}{\Delta x_i} \tag{3}$$

If two variables have widely separated speeds, they are candidates for time-scale separation.

Since it is the maximum attainable values of the speeds that are of comparative intererst, one possibility for computing the speeds (method 1) is

$$S_{1_i} = \frac{1}{\Delta x_i} \max_{\substack{u\in U \\ x\in[x_m,x_M]}} f_i(x,u) \tag{4}$$

This has the advantage that the assessment of variable speed is global; however, there is the disadvantage that the resulting system states may be unrealistic, as will be demonstated by the example in a subsequent section.

Another possibility (method 2) is to select a value of the state in the region of interest, $\bar{x}$,

$$S_{2_i} = \frac{1}{\Delta x_i} \max_{u\in U} f_i(\bar{x},u) \tag{5}$$

This equation has the obvious, usually undesirable, feature of depending on the value of $\bar{x}$ selected. Both S_{1_i} and S_{2_i} have the desirable properties that they are invariant under linear transformation of state variables $y = ax + b$ and that they have the same dimensions for all variables.

It is important to realize that a wide separation of speed as computed by Eq. (4) or (5) is not sufficient to insure a satisfactory singular perturbation formulation since state variable coupling may also be important. This has been emphasized for nonlinear systems in Refs. 4 and 24 and is explicit in the relations giving time-scale separation requirements for linear systems.[21,22]

Aircraft Trajectory Optimization Problem

The usual dynamical equations used in trajectory analysis of high-performance aircraft are those of a point mass[5]

$$\begin{aligned} \dot{x} &= V\cos\gamma\cos\chi & \dot{V} &= g(T - D - \sin\gamma) \\ \dot{y} &= V\cos\gamma\sin\chi & \dot{\chi} &= gL\sin\mu / V\cos\gamma \\ \dot{h} &= V\sin\gamma & \dot{\gamma} &= g(L\cos\mu - \cos\gamma)/V \end{aligned} \tag{6}$$

The thrust and drag per unit weight are given by

$$T = \beta T_M(h,V), \qquad D = D_0(h,V) + D_i(h,V)L^2 \tag{7}$$

The constraints on the control variables β (throttle setting), μ (bank angle), and L (lift per unit weight) are

$$\begin{gathered} 0 \le \beta \le 1 \\ -\pi \le \mu \le \pi \\ L_m \le L \le L_M = \min[n, \hat{L}(h,V)] \end{gathered} \tag{8}$$

where n is the maximum (structural) load factor and $\hat{L}$ the lift as limited by maximum angle of attack, α_M,

$$\hat{L}(h,V) = C_{L_\alpha}(h,V)\alpha_M \rho(h) V^2 S/2W \tag{9}$$

where C_{L_α} is the lift curve slope, ρ the atmospheric density, and S the wing reference area.

A new variable, the specific energy E, defined by

$$E = h + (1/2g)V^2 \tag{10}$$

is often introduced; it can be used in place of either h or V as a state variable. From Eqs. (6) and (10), the state equation for E is

$$\dot{E} = V(T - D) \tag{11}$$

As mentioned previously, almost all analyses of Eq. (6) by singular perturbation methods have consisted of artificial insertion of small parameters based on ad hoc time-scale judgments. Several of these formulations are reviewed in Table 1. Although there is diversity in the details of the ordering, generally (x,y) have been treated as the slowest variables, (E,χ) as the intermediate variables, and (h,γ) as the fastest varaibles. Note that all analyses (except those for problems in a horizontal plane) have used E in place of V as a state variable. This is precisely because h and V typically are of the same speed, whereas E is slower than h and V; thus, the use of E and h allows a time-scale separation of E and h.

The speed estimates Eqs. (4) and (5) are now applied to the state equations (6), see Table 2. The formula for S_{1_γ} contains the corner velocity $V_c(h)$, or speed for highest turn rate, and is based on the assumption that inverted flight is not allowed. Relative to method 2 [Eq. (5)], method 1 [Eq. (4)] has the disadvantage of giving unrealistic flight conditions. Four of the method 1 conditions are for vertical flight and some of these will be unsustainable or even unobtainable for most aircraft, such as, for example, the conditions for S_{1_h}. Also, the speed of χ is infinite. On the other hand, the speed estimates of method 2 are highly sensitive to location in state space. For example, if $\bar{\gamma} = 0$ is selected, the speed of h is zero. Also, the highly nonlinear nature of the functions $T_M(h,V)$ and $D_0(h,V)$ makes the selection of $\bar{h}$ and $\bar{V}$ critical.

Numerical Example

For a numerical example, we choose the F-4C aircraft. For this aircraft, $\Delta h = 25{,}300$ m, $\Delta V = 509$ m/s, $\Delta E = 43{,}000$ m, $W = 15{,}900$ kg, $n = 6$, $S = 49.2$ m^2, and $\alpha_M = 12$ deg. We also take $\sqrt{\Delta x^2 + \Delta y^2} = R = 12.2\times 10^5$ m, $\Delta\chi = \pi$, and $\Delta\gamma = \pi$. Analysis of the aircraft data gives

$$\min_h V_c = 207 \text{ m/s at } h = 0$$

$$\max\ (T_M - D_0) = 0.82 \text{ at } h = 0, \quad \underset{h,V}{V = 289} \text{ m/s}$$

$$\max_{h,V}\ (T_M - D_0) = 0.82 \text{ at } h = 0, \quad \underset{h,V}{V = 289} \text{ m/s}$$

For method 2, we take $\bar{h} = 6100$ m, $\bar{V} = 244$ m/s, $\bar{\chi} = 45$ deg, and $\bar{\gamma} = 17.5$ deg, giving

$$\bar{T}_M = T_M(\bar{h},\bar{V}) = 0.554$$

$$\bar{D}_0 = D_0(\bar{h},\bar{V}) = 0.080$$

$$\bar{L}_M = \min[n, \hat{L}(\bar{h},\bar{V})] = 4.42$$

$$\bar{E} = \bar{h} + (1/2g)\bar{V}^2 = 9110 \text{ m}$$

Using these data in the relations of Table 2 gives the results shown in Table 3. Both methods give the same general order of variable speeds from slowest to fastest: (x,y), E, h, V, γ, χ. However, there is a great deal of difference in the

Table 3 Estimates of state variable speed for F-4C aircraft

Variable	Speed	
	Method 1	Method 2
x,y	0.00049	0.00013
E	0.0055	0.0027
h	0.023	0.0029
V	0.035	0.0033
γ	0.090	0.044
χ	∞	0.059

magnitude of the time-scale separations. For example, method 1 predicts that E is much slower than h or V, which are themselves of about the same speed, whereas method 2 indicates that E, h, and V all have very nearly the same speed.

Comparing the variable ordering shown in Table 3 with the ordering assumed in past analyses (Table 1) shows general agreement. The only exception is that χ has been treated as a variable of intermediate speed, whereas the present analysis shows it to be the fastest variable of all. This discrepancy has been recognized,[8-19] but χ has been retained as an intermediate variable for two main reasons in spite of this recognition. First, treating χ as slower than h and γ gives singular perturbation solutions that model maneuvers such as "high- and low-speed yo-yos," which are known to be important in optimal turning of high-performance aircraft, whereas treating χ as faster than h and γ does not.[12] Second, the adjoint equation associated with χ can be analytically integrated, making the inclusion of χ in slower subsystems relatively easy.

Conclusion

Two methods for time-scale separation analysis of dynamic systems have been proposed. These methods are based on the concept of state variable speed and require knowledge only of the dynamical equations and bounds on state and control variables. They are not as rigorous as other proposed methods, but they do not require a priori knowledge of an optimal trajectory, are relatively easy to apply, and are an improvement over the ad hoc methods currently in use.

The two methods were applied to a typical class of aircraft flight dynamics problems and equations were derived for state variable speed estimation. A numerical example showed that the time-scale separations as computed by the two methods proposed here generally agree with previous ad hoc time-scale separation assumptions.

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Optimal Finite Horizon Approximation of Unstable Linear Systems

A.-M. Guillaume*
Université Catholique de Louvain
Louvain-la-Neuve, Belgium
and
P.T. Kabamba
University of Michigan, Ann Arbor, Michigan

Introduction

THE general problem of approximating a high-order linear dynamical system by a low-order "reduced-order model" has received considerable attention in the literature during the last fifteen years. See Ref. (1) for an extensive bibliographical

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*Research Assistant, Unité de Mécanique Appliquée.
†Assistant Professor, Department of Aerospace Engineering.